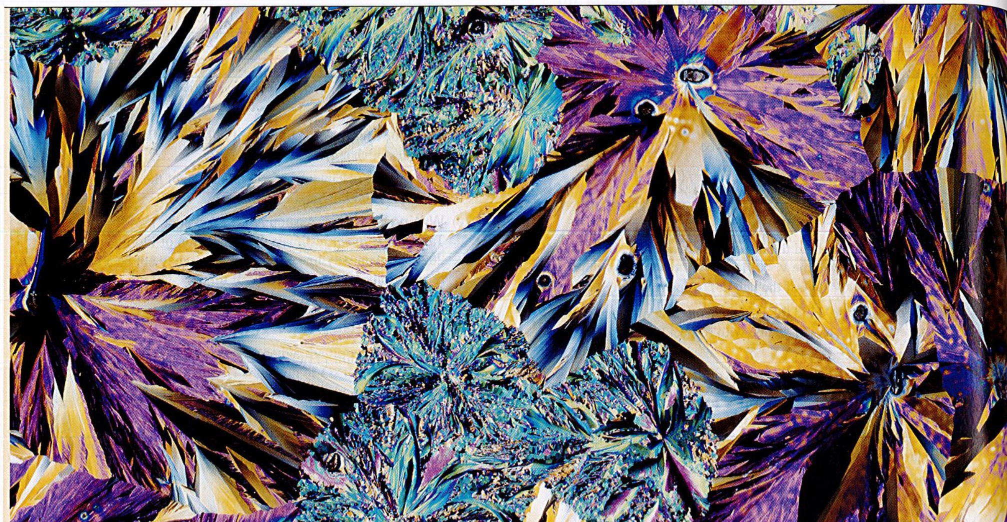
CHAPTER

3 Derivatives



Shown here is the pain reliever acetaminophen in crystalline form, photographed under a transmitted light microscope. While acetaminophen relieves pain with few side effects, it is toxic in large doses. One study found that only 30% of parents who gave acetaminophen to their children could accurately calculate and measure the correct dose.

One rule for calculating the dosage (mg) of acetaminophen for children ages 1 to 12 years old is $D(t) = 750t/(t + 12)$, where t is age in years. What is an expression for the rate of change of a child's dosage with respect to the child's age? How does the rate of change of the dosage relate to the growth rate of children? This problem can be solved with the information covered in Section 3.4.

CHAPTER 3 Overview

In Chapter 2, we learned how to find the slope of a tangent to a curve as the limit of the slopes of secant lines. In Example 4 of Section 2.4, we derived a formula for the slope of the tangent at an arbitrary point $(a, 1/a)$ on the graph of the function $f(x) = 1/x$ and showed that it was $-1/a^2$.

This seemingly unimportant result is more powerful than it might appear at first glance, as it gives us a simple way to calculate the instantaneous rate of change of f at any point. The study of rates of change of functions is called *differential calculus,* and the formula $-1/a^2$ was our first look at a *derivative.* The derivative was the 17th-century breakthrough that enabled mathematicians to unlock the secrets of planetary motion and gravitational attraction—of objects changing position over time. We will learn many uses for derivatives in Chapter 5, but first, in the next two chapters, we will focus on what derivatives are and how they work.

3.1 Derivative of a Function

What you will learn about . . .

- Definition of Derivative
- Notation
- Relationships Between the Graphs of f and f'
- Graphing the Derivative from Data
- One-sided Derivatives

and why . . .

The derivative is the key to modeling instantaneous change mathematically.

Definition of Derivative

In Section 2.4, we defined the slope of a curve $y = f(x)$ at the point where $x = a$ to be

$$m = \lim_{h\to 0} \frac{f(a + h) - f(a)}{h}.$$

When it exists, this limit is called the **derivative of f at a.** In this section, we investigate the derivative as a *function* derived from f by considering the limit at each point of the domain of f.

DEFINITION Derivative

The **derivative** of the function f with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h\to 0} \frac{f(x + h) - f(x)}{h}, \quad (1)$$

provided the limit exists.

The domain of f', the set of points in the domain of f for which the limit exists, may be smaller than the domain of f. If $f'(x)$ exists, we say that f **has a derivative (is differentiable)** at x. A function that is differentiable at every point of its domain is a **differentiable function.**

EXAMPLE 1 Applying the Definition

Differentiate (that is, find the derivative of) $f(x) = x^3$.

continued

SOLUTION

Applying the definition, we have

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{(x+h)^3-x^3}{h} && \text{Eq. 1 with } f(x)=x^3,\ f(x+h)=(x+h)^3 \\
&= \lim_{h\to 0}\frac{(x^3+3x^2h+3xh^2+h^3)-x^3}{h} && (x+h)^3 \text{ expanded} \\
&= \lim_{h\to 0}\frac{(3x^3+3xh+h^2)h}{h} && x^3 \text{ terms cancelled, } h \text{ factored out} \\
&= \lim_{h\to 0}(3x^2+3xh+h^2) = 3x^2.
\end{aligned}
$$

Now Try Exercise 1.

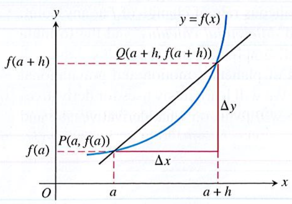


Figure 3.1 The slope of the secant line PQ is

$$\frac{\Delta y}{\Delta x} = \frac{f(a+h)-f(a)}{(a+h)-a} = \frac{f(a+h)-f(a)}{h}.$$

The derivative of $f(x)$ at a point where $x = a$ is found by taking the limit as $h \to 0$ of slopes of secant lines, as shown in Figure 3.1.

By relabeling the picture as in Figure 3.2, we arrive at a useful alternate formula for calculating the derivative. This time, the limit is taken as x approaches a.

DEFINITION (ALTERNATE) Derivative at a Point

The **derivative** of the function f **at the point** $x = a$ is the limit

$$f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}, \qquad (2)$$

provided the limit exists.

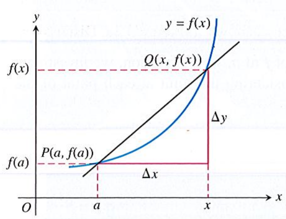


Figure 3.2 The slope of the secant line PQ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x)-f(a)}{x-a}.$$

After we find the derivative of f at a point $x = a$ using the alternate form, we can find the derivative of f as a function by applying the resulting formula to an arbitrary x in the domain of f.

EXAMPLE 2 Applying the Alternate Definition

Differentiate $f(x) = \sqrt{x}$ using the alternate definition.

SOLUTION

At the point $x = a$,

$$
\begin{aligned}
f'(a) &= \lim_{x\to a}\frac{f(x)-f(a)}{x-a} \\
&= \lim_{x\to a}\frac{\sqrt{x}-\sqrt{a}}{x-a} && \text{Eq. 2 with } f(x)=\sqrt{x} \\
&= \lim_{x\to a}\frac{\sqrt{x}-\sqrt{a}}{x-a}\cdot\frac{\sqrt{x}+\sqrt{a}}{\sqrt{x}+\sqrt{a}} && \text{Rationalize\ldots} \\
&= \lim_{x\to a}\frac{x-a}{(x-a)(\sqrt{x}+\sqrt{a})} && \text{\ldots the numerator.} \\
&= \lim_{x\to a}\frac{1}{\sqrt{x}+\sqrt{a}} && \text{We can now take the limit.} \\
&= \frac{1}{2\sqrt{a}}.
\end{aligned}
$$

Applying this formula to an arbitrary $x > 0$ in the domain of f identifies the derivative as the function $f'(x) = 1/(2\sqrt{x})$ with domain $(0, \infty)$. ***Now Try Exercise 5.***

Why all the notation?

The "prime" notations y' and f' come from notations that Newton used for derivatives. The d/dx notations are similar to those used by Leibniz. Each has its advantages and disadvantages.

Notation

There are many ways to denote the derivative of a function $y = f(x)$. Besides $f'(x)$, the most common notations are these:

y'	"y prime"	Nice and brief, but does not name the independent variable.
$\dfrac{dy}{dx}$	"dy dx" or "the derivative of y with respect to x"	Names both variables and uses d for derivative.
$\dfrac{df}{dx}$	"df dx" or "the derivative of f with respect to x"	Emphasizes the function's name.
$\dfrac{d}{dx}f(x)$	"d dx of f at x" or "the derivative of f at x"	Emphasizes the idea that differentiation is an operation performed on f.

Relationships Between the Graphs of f and f'

When we have the explicit formula for $f(x)$, we can derive a formula for $f'(x)$ using methods like those in Examples 1 and 2. We have already seen, however, that functions are encountered in other ways: graphically, for example, or in tables of data.

Because we can think of the derivative at a point in graphical terms as *slope*, we can get a good idea of what the graph of the function f' looks like by *estimating the slopes* at various points along the graph of f.

EXAMPLE 3 Graphing f' from f

Graph the derivative of the function f whose graph is shown in Figure 3.3a. Discuss the behavior of f in terms of the signs and values of f'.

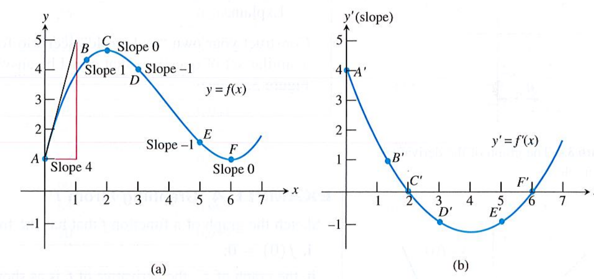


Figure 3.3 By plotting the slopes at points on the graph of $y = f(x)$, we obtain a graph of $y' = f'(x)$. The slope at point A of the graph of f in part (a) is the y-coordinate of point A' on the graph of f' in part (b), and so on. (Example 3)

SOLUTION

First, we draw a pair of coordinate axes, marking the horizontal axis in x-units and the vertical axis in slope units (Figure 3.3b). Next, we estimate the slope of the graph of f at various points, plotting the corresponding slope values using the new axes. At $A(0, f(0))$, the graph of f has slope 4, so $f'(0) = 4$. At B, the graph of f has slope 1, so $f' = 1$ at B', and so on.

continued

We complete our estimate of the graph of f' by connecting the plotted points with a smooth curve.

Although we do not have a formula for either f or f', the graph of each reveals important information about the behavior of the other. In particular, notice that f is decreasing where f' is negative and increasing where f' is positive. Where f' is zero, the graph of f has a horizontal tangent, changing from increasing to decreasing at point C and from decreasing to increasing at point F. ***Now Try Exercise 23.***

EXPLORATION 1 Reading the Graphs

Suppose that the function f in Figure 3.3a represents the depth y (in inches) of water in a ditch alongside a dirt road as a function of time x (in days). How would you answer the following questions?

1. What does the graph in Figure 3.3b represent? What units would you use along the y'-axis?
2. Describe as carefully as you can what happened to the water in the ditch over the course of the 7-day period.
3. Can you describe the weather during the 7 days? When was it the wettest? When was it the driest?
4. How does the graph of the derivative help in finding when the weather was wettest or driest?
5. Interpret the significance of point C in terms of the water in the ditch. How does the significance of point C' reflect that in terms of rate of change?
6. It is tempting to say that it rains right up until the beginning of the second day, but that overlooks a fact about rainwater that is important in flood control. Explain.

Construct your own "real-world" scenario for the function in Example 3, and pose a similar set of questions that could be answered by considering the two graphs in Figure 3.3.

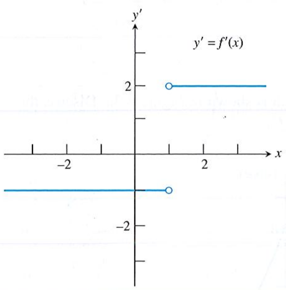


Figure 3.4 The graph of the derivative. (Example 4)

EXAMPLE 4 Graphing f from f'

Sketch the graph of a function f that has the following properties:

i. $f(0) = 0$;

ii. the graph of f', the derivative of f, is as shown in Figure 3.4;

iii. f is continuous for all x.

SOLUTION

To satisfy property (i), we begin with a point at the origin.

To satisfy property (ii), we consider what the graph of the derivative tells us about slopes. To the left of $x = 1$, the graph of f has a constant slope of -1; therefore we draw a line with slope -1 to the left of $x = 1$, making sure that it goes through the origin.

To the right of $x = 1$, the graph of f has a constant slope of 2, so it must be a line with slope 2. There are infinitely many such lines but only one—the one that meets the left side of the graph at $(1, -1)$—will satisfy the continuity requirement. The resulting graph is shown in Figure 3.5. ***Now Try Exercise 27.***

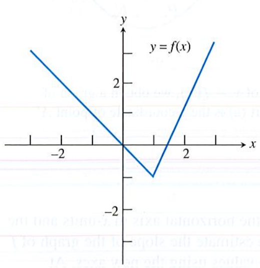
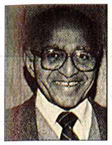


Figure 3.5 The graph of f, constructed from the graph of f' and two other conditions. (Example 4)

What's happening at $x = 1$?

Notice that f in Figure 3.5 is defined at $x = 1$, while f' is not. It is the continuity of f that enables us to conclude that $f(1) = -1$. Looking at the graph of f, can you see why f' could not possibly be defined at $x = 1$? We will explore the reason for this in Example 6.

David H. Blackwell (1919–2010)



By the age of 22, David Blackwell had earned a Ph.D. in Mathematics from the University of Illinois. He taught at Howard University, where his research included statistics, Markov chains, and sequential analysis. He then went on to teach and continue his research at the University of California at Berkeley. Dr. Blackwell served as president of the American Statistical Association and was the first African American mathematician of the National Academy of Sciences.

Graphing the Derivative from Data

Discrete points plotted from sets of data do not yield a continuous curve, but we have seen that the shape and pattern of the graphed points (called a scatter plot) can be meaningful nonetheless. It is often possible to fit a curve to the points using regression techniques. If the fit is good, we could use the curve to get a graph of the derivative visually, as in Example 3. However, it is also possible to get a scatter plot of the derivative numerically, directly from the data, by computing the slopes between successive points, as in Example 5.

EXAMPLE 5 Estimating the Probability of Shared Birthdays

Suppose 30 people are in a room. What is the probability that two of them share the same birthday? Ignore the year of birth.

SOLUTION

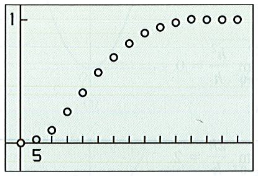
It may surprise you to learn that the probability of a shared birthday among 30 people is at least 0.706, well above two-thirds! In fact, if we assume that no one day is more likely to be a birthday than any other day, the probabilities shown in Table 3.1 are not hard to determine (see Exercise 45).

TABLE 3.1 Probabilities of Shared Birthdays

People in Room (x)	Probability (y)
0	0
5	0.027
10	0.117
15	0.253
20	0.411
25	0.569
30	0.706
35	0.814
40	0.891
45	0.941
50	0.970
55	0.986
60	0.994
65	0.998
70	0.999

TABLE 3.2 Estimates of Slopes on the Probability Curve

Midpoint of Interval (x)	Change (Slope $\Delta y/\Delta x$)
2.5	0.0054
7.5	0.0180
12.5	0.0272
17.5	0.0316
22.5	0.0316
27.5	0.0274
32.5	0.0216
37.5	0.0154
42.5	0.0100
47.5	0.0058
52.5	0.0032
57.5	0.0016
62.5	0.0008
67.5	0.0002



[–5, 75] by [–0.2, 1.1]

Figure 3.6 Scatter plot of the probabilities (y) of shared birthdays among x people, for $x = 0, 5, 10, \ldots, 70$. (Example 5)

A scatter plot of the data in Table 3.1 is shown in Figure 3.6.

Notice that the probabilities grow slowly at first, then faster, then much more slowly past $x = 45$. At which x are they growing the fastest? To answer the question, we need the graph of the derivative.

Using the data in Table 3.1, we compute the slopes between successive points on the probability plot. For example, from $x = 0$ to $x = 5$ the slope is

$$\frac{0.027 - 0}{5 - 0} = 0.0054.$$

We make a new table showing the slopes, beginning with slope 0.0054 on the interval $[0, 5]$ (Table 3.2). A logical x-value to use to represent the interval is its midpoint, 2.5.

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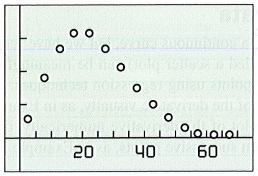


[–5, 75] by [–0.01, 0.04]

Figure 3.7 A scatter plot of the derivative data in Table 3.2. (Example 5)

A scatter plot of the derivative data in Table 3.2 is shown in Figure 3.7.

From the derivative plot, we can see that the rate of change peaks near $x = 20$. You can impress your friends with your "psychic powers" by predicting a shared birthday in a room of just 25 people (since you will be right about 57% of the time), but the derivative warns you to be cautious: A few less people can make quite a difference. On the other hand, going from 40 people to 100 people will not improve your chances much at all.

Now Try Exercise 29.

Generating shared birthday probabilities: If you know a little about probability, you might try generating the probabilities in Table 3.1. Extending the Idea Exercise 45 at the end of this section shows how to generate them on a calculator.

One-Sided Derivatives

A function $y = f(x)$ is **differentiable on a closed interval** $[a, b]$ if it has a derivative at every interior point of the interval, and if the limits

$$\lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h} \qquad \text{[the \textbf{right-hand derivative at } a\text{]}}$$

$$\lim_{h \to 0^-} \frac{f(b+h) - f(b)}{h} \qquad \text{[the \textbf{left-hand derivative at } b\text{]}}$$

exist at the endpoints. In the right-hand derivative, h is positive and $a + h$ approaches a from the right. In the left-hand derivative, h is negative and $b + h$ approaches b from the left (Figure 3.8).

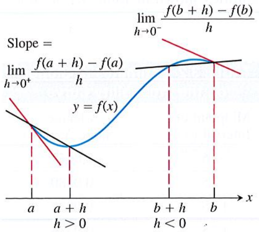


Figure 3.8 Derivatives at endpoints are one-sided limits.

Right-hand and left-hand derivatives may be defined at any point of a function's domain.

The usual relationship between one-sided and two-sided limits holds for derivatives. Theorem 3, Section 2.1, allows us to conclude that a function has a (two-sided) derivative at a point if and only if the function's right-hand and left-hand derivatives are defined and equal at that point.

EXAMPLE 6 One-Sided Derivatives Can Differ at a Point

Show that the following function has left-hand and right-hand derivatives at $x = 0$, but no derivative there (Figure 3.9).

$$y = \begin{cases} x^2, & x \le 0 \\ 2x, & x > 0 \end{cases}$$

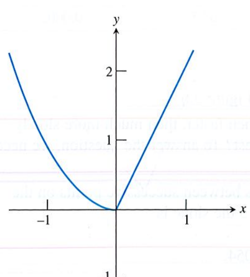


Figure 3.9 A function with different one-sided derivatives at $x = 0$. (Example 6)

SOLUTION

We verify the existence of the left-hand derivative:

$$\lim_{h \to 0^-} \frac{(0+h)^2 - 0^2}{h} = \lim_{h \to 0^-} \frac{h^2}{h} = 0.$$

We verify the existence of the right-hand derivative:

$$\lim_{h \to 0^+} \frac{2(0+h) - 0^2}{h} = \lim_{h \to 0^+} \frac{2h}{h} = 2.$$

Since the left-hand derivative equals zero and the right-hand derivative equals 2, the derivatives are not equal at $x = 0$. The function does not have a derivative at 0.

Now Try Exercise 31.

Quick Review 3.1 (For help, go to Sections 2.1 and 2.4.)

Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator.*

In Exercises 1–4, evaluate the indicated limit algebraically.

1. $\lim_{h \to 0} \frac{(2+h)^2 - 4}{h}$

2. $\lim_{x \to 2^+} \frac{x+3}{2}$

3. $\lim_{y \to 0^-} \frac{|y|}{y}$

4. $\lim_{x \to 4} \frac{2x - 8}{\sqrt{x} - 2}$

5. Find the slope of the line tangent to the parabola $y = x^2 + 1$ at its vertex.

6. By considering the graph of $f(x) = x^3 - 3x^2 + 2$, find the intervals on which f is increasing.

In Exercises 7–10, let

$$f(x) = \begin{cases} x + 2, & x \le 1 \\ (x-1)^2, & x > 1. \end{cases}$$

7. Find $\lim_{x \to 1^+} f(x)$ and $\lim_{x \to 1^-} f(x)$.

8. Find $\lim_{h \to 0^+} f(1 + h)$.

9. Does $\lim_{x \to 1} f(x)$ exist? Explain.

10. Is f continuous? Explain.

Section 3.1 Exercises

In Exercises 1–4, use the definition

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

to find the derivative of the given function at the given value of a.

1. $f(x) = 1/x, a = 2$

2. $f(x) = x^2 + 4, a = 1$

3. $f(x) = 3 - x^2, a = -1$

4. $f(x) = x^3 + x, a = 0$

In Exercises 5–8, use the definition

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

to find the derivative of the given function at the given value of a.

5. $f(x) = 1/x, a = 2$

6. $f(x) = x^2 + 4, a = 1$

7. $f(x) = \sqrt{x+1}, a = 3$

8. $f(x) = 2x + 3, a = -1$

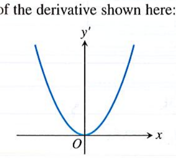
9. Find $f'(x)$ if $f(x) = 3x - 12$.

10. Find dy/dx if $y = 7x$.

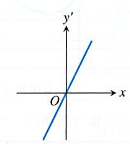
11. Find $\frac{d}{dx}(x^2)$.

12. Find $\frac{d}{dx} f(x)$ if $f(x) = 3x^2$.

In Exercises 13–16, match the graph of the function with the graph of the derivative shown here:



(a)



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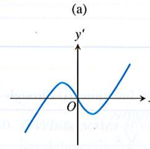

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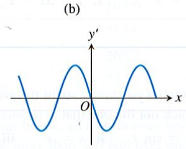

(d)

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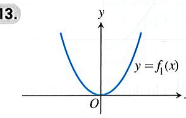

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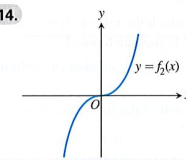

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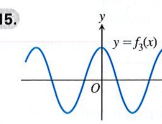
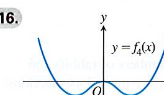


16.



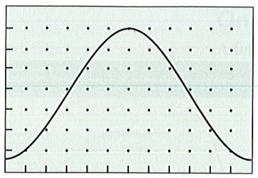
17. If $f(2) = 3$ and $f'(2) = 5$, find an equation of **(a)** the *tangent* line, and **(b)** the *normal* line to the graph of $y = f(x)$ at the point where $x = 2$.
[*Hint:* Recall that the normal line is perpendicular to the tangent line.]

18. Find the derivative of the function $y = 2x^2 - 13x + 5$ and use it to find an equation of the line tangent to the curve at $x = 3$.

19. Find the lines that are **(a)** tangent and **(b)** normal to the curve $y = x^3$ at the point $(1, 1)$.

20. Find the lines that are (a) tangent and **(b)** normal to the curve $y = \sqrt{x}$ at $x = 4$.

21. ***Daylight in Fairbanks*** The viewing window below shows the number of hours of daylight in Fairbanks, Alaska, on each day for a typical 365-day period from January 1 to December 31. Answer the following questions by estimating slopes on the graph in hours per day. For the purposes of estimation, assume that each month has 30 days.



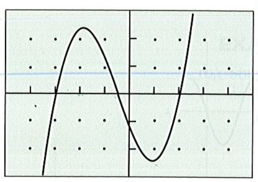
[0, 365] by [0, 24]

(a) On about what date is the amount of daylight increasing at the fastest rate? What is that rate?

(b) Do there appear to be days on which the rate of change in the amount of daylight is zero? If so, which ones?

(c) On what dates is the rate of change in the number of daylight hours positive? negative?

22. ***Graphing f′ from f*** Given the graph of the function f below, sketch a graph of the *derivative* of f.



[–5, 5] by [–3, 3]

23. The graphs in Figure 3.10a show the numbers of rabbits and foxes in a small arctic population. They are plotted as functions of time for 200 days. The number of rabbits increases at first, as the rabbits reproduce. But the foxes prey on the rabbits and, as the number of foxes increases, the rabbit population levels off and then drops. Figure 3.10b shows the graph of the derivative of the rabbit population. We made it by plotting slopes, as in Example 3.

(a) What is the value of the derivative of the rabbit population in Figure 3.10 when the number of rabbits is largest? smallest?

(b) What is the size of the rabbit population in Figure 3.10 when its derivative is largest? smallest?

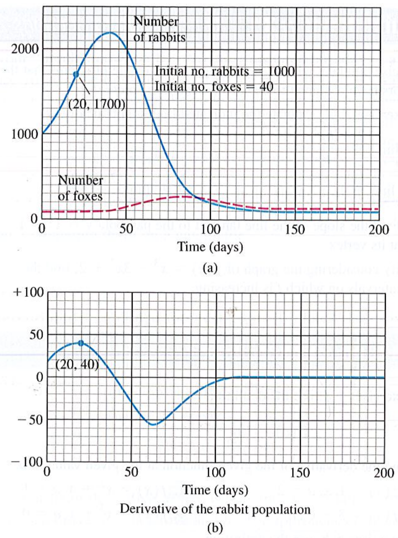
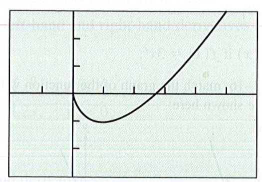


Figure 3.10 Rabbits and foxes in an arctic predator-prey food chain. *Source: Differentiation* by W. U. Walton et al., Project CALC, Education Development Center, Inc., Newton, MA, 1975, p. 86.

24. Shown below is the graph of $f(x) = x \ln x - x$. From what you know about the graphs of functions (i) through (v), pick out the one that is the *derivative* of f for $x > 0$.



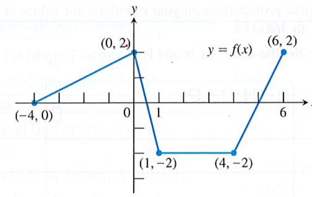
[–2, 6] by [–3, 3]

i. $y = \sin x$ **ii.** $y = \ln x$ **iii.** $y = \sqrt{x}$
iv. $y = x^2$ **v.** $y = 3x - 1$

25. From what you know about the graphs of functions (i) through (v), pick out the one that is *its own derivative.*

i. $y = \sin x$ **ii.** $y = x$ **iii.** $y = \sqrt{x}$
iv. $y = e^x$ **v.** $y = x^2$

26. The graph of the function $y = f(x)$ shown here is made of line segments joined end to end.



(a) Graph the function's derivative.

(b) At what values of x between $x = -4$ and $x = 6$ is the function not differentiable?

27. ***Graphing f from f′*** Sketch the graph of a continuous function f with $f(0) = -1$ and

$$f'(x) = \begin{cases} 1, & x < -1 \\ -2, & x > -1. \end{cases}$$

28. ***Graphing f from f′*** Sketch the graph of a continuous function f with $f(0) = 1$ and

$$f'(x) = \begin{cases} 2, & x < 2 \\ -1, & x > 2. \end{cases}$$

In Exercises 29 and 30, use the data to answer the questions.

29. ***A Downhill Skier*** Table 3.3 gives the approximate distance traveled by a downhill skier after t seconds for $0 \le t \le 10$. Use the method of Example 5 to sketch a graph of the derivative; then answer the following questions:

(a) What does the derivative represent?

(b) In what units would the derivative be measured?

(c) Can you guess an equation of the derivative by considering its graph?

TABLE 3.3 Skiing Distances

Time t (seconds)	Distance Traveled (feet)
0	0
1	3.3
2	13.3
3	29.9
4	53.2
5	83.2
6	119.8
7	163.0
8	212.9
9	269.5
10	332.7

30. ***A Whitewater River*** Bear Creek, a Georgia river known to kayaking enthusiasts, drops more than 770 feet over one stretch of 3.24 miles. By reading a contour map, one can estimate the elevations (y) at various distances (x) downriver from the start of the kayaking route (Table 3.4).

TABLE 3.4 Elevations Along Bear Creek

Distance Downriver (miles)	River Elevation (feet)
0.00	1577
0.56	1512
0.92	1448
1.19	1384
1.30	1319
1.39	1255
1.57	1191
1.74	1126
1.98	1062
2.18	998
2.41	933
2.64	869
3.24	805

(a) Sketch a graph of elevation (y) as a function of distance downriver (x).

(b) Use the technique of Example 5 to get an approximate graph of the derivative, dy/dx.

(c) The average change in elevation over a given distance is called a *gradient.* In this problem, what units of measure would be appropriate for a gradient?

(d) In this problem, what units of measure would be appropriate for the derivative?

(e) How would you identify the most dangerous section of the river (ignoring rocks) by analyzing the graph in (a)? Explain.

(f) How would you identify the most dangerous section of the river by analyzing the graph in (b)? Explain.

31. Using one-sided derivatives, show that the function

$$f(x) = \begin{cases} x^2 + x, & x \le 1 \\ 3x - 2, & x > 1 \end{cases}$$

does not have a derivative at $x = 1$.

32. Using one-sided derivatives, show that the function

$$f(x) = \begin{cases} x^3, & x \le 1 \\ 3x, & x > 1 \end{cases}$$

does not have a derivative at $x = 1$.

33. Writing to Learn Graph $y = \sin x$ and $y = \cos x$ in the same viewing window. Which function could be the derivative of the other? Defend your answer in terms of the behavior of the graphs.

34. In Example 2 of this section we showed that the derivative of $y = \sqrt{x}$ is a function with domain $(0, \infty)$. However, the function $y = \sqrt{x}$ itself has domain $[0, \infty)$, so it could have a right-hand derivative at $x = 0$. Prove that it does not.

35. Writing to Learn Use the concept of the derivative to define what it might mean for two parabolas to be parallel. Construct equations for two such parallel parabolas and graph them. Are the parabolas "everywhere equidistant," and if so, in what sense?

Standardized Test Questions

36. True or False If $f(x) = x^2 + x$, then $f'(x)$ exists for every real number x. Justify your answer.

37. True or False If the left-hand derivative and the right-hand derivative of f exist at $x = a$, then $f'(a)$ exists. Justify your answer.

38. Multiple Choice Let $f(x) = 4 - 3x$. Which of the following is equal to $f'(-1)$?

(A) −7 (B) 7 (C) −3 (D) 3 (E) does not exist

39. Multiple Choice Let $f(x) = 1 - 3x^2$. Which of the following is equal to $f'(1)$?

(A) −6 (B) −5 (C) 5 (D) 6 (E) does not exist

In Exercises 40 and 41, let

$$f(x) = \begin{cases} x^2 - 1, & x < 0 \\ 2x - 1, & x \ge 0. \end{cases}$$

40. Multiple Choice Which of the following is equal to the left-hand derivative of f at $x = 0$?

(A) −2 (B) 0 (C) 2 (D) ∞ (E) $-\infty$

41. Multiple Choice Which of the following is equal to the right-hand derivative of f at $x = 0$?

(A) −2 (B) 0 (C) 2 (D) ∞ (E) $-\infty$

Explorations

42. Let $f(x) = \begin{cases} x^2, & x \le 1 \\ 2x, & x > 1. \end{cases}$

(a) Find $f'(x)$ for $x < 1$. (b) Find $f'(x)$ for $x > 1$.

(c) Find $\lim_{x \to 1^-} f'(x)$. (d) Find $\lim_{x \to 1^+} f'(x)$.

(e) Does $\lim_{x \to 1} f'(x)$ exist? Explain.

(f) Use the definition to find the left-hand derivative of f at $x = 1$ if it exists.

(g) Use the definition to find the right-hand derivative of f at $x = 1$ if it exists.

(h) Does $f'(1)$ exist? Explain.

43. Group Activity Using graphing calculators, have each person in your group do the following:

(a) pick two numbers a and b between 1 and 10;

(b) graph the function $y = (x - a)(x + b)$;

(c) graph the *derivative* of your function (it will be a line with slope 2);

(d) find the y-intercept of your derivative graph.

(e) Compare your answers and determine a simple way to predict the y-intercept, given the values of a and b. Test your result.

Extending the Ideas

44. Find the unique value of k that makes the function

$$f(x) = \begin{cases} x^3, & x \le 1 \\ 3x + k, & x > 1 \end{cases}$$

differentiable at $x = 1$.

45. *Generating the Birthday Probabilities* Example 5 of this section concerns the probability that, in a group of n people, at least two people will share a common birthday. You can generate these probabilities on your calculator for values of n from 1 to 365.

Step 1: Set the values of N and P to zero:

```
0→N:0→P
                    0
```

Step 2: Type in this single, multi-step command:

```
N+1→N:1−(1−P)(366
−N)/365→P:{N,P}
```

Now each time you press the ENTER key, the command will print a new value of N (the number of people in the room) alongside P (the probability that at least two of them share a common birthday):

```
                {1 0}
      {2 .002739726}
      {3 .0082041659}
      {4 .0163559125}
      {5 .0271355737}
      {6 .0404624836}
      {7 .0562357031}
```

If you have some experience with probability, try to answer the following questions without looking at the table:

(a) If there are three people in the room, what is the probability that they all have *different* birthdays? (Assume that there are 365 possible birthdays, all of them equally likely.)

(b) If there are three people in the room, what is the probability that at least two of them share a common birthday?

(c) Explain how you can use the answer in part (b) to find the probability of a shared birthday when there are *four* people in the room. (This is how the calculator statement in Step 2 generates the probabilities.)

(d) Is it reasonable to assume that all calendar dates are equally likely birthdays? Explain your answer.

3.2 Differentiability

What you will learn about . . .

- How $f'(a)$ Might Fail to Exist
- Differentiability Implies Local Linearity
- Numerical Derivatives on a Calculator
- Differentiability Implies Continuity
- Intermediate Value Theorem for Derivatives

and why . . .

Graphs of differentiable functions can be approximated by their tangent lines at points where the derivative exists.

How rough can the graph of a continuous function be?

The graph of the absolute value function fails to be differentiable at a single point. If you graph $y = \sin^{-1}(\sin(x))$ on your calculator, you will see a continuous function with an *infinite* number of points of nondifferentiability. But can a continuous function fail to be differentiable at *every* point?

The answer, surprisingly enough, is yes, as Karl Weierstrass showed in 1872. One of his formulas (there are many like it) was

$$f(x) = \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n \cos(9^n \pi x),$$

a formula that expresses f as an infinite (but converging) sum of cosines with increasingly higher frequencies. By adding wiggles to wiggles infinitely many times, so to speak, the formula produces a function whose graph is too bumpy in the limit to have a tangent anywhere!

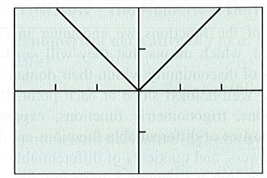
How $f'(a)$ Might Fail to Exist

A function will not have a derivative at a point $P(a, f(a))$ where the slopes of the secant lines,

$$\frac{f(x) - f(a)}{x - a},$$

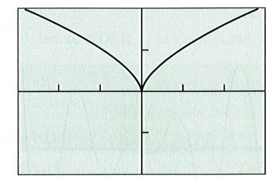
fail to approach a limit as x approaches a. Figures 3.11–3.14 illustrate four different instances where this occurs. For example, a function whose graph is otherwise smooth will fail to have a derivative at a point where the graph has

1. a *corner,* where the one-sided derivatives differ; Example: $f(x) = |x|$



[−3, 3] by [−2, 2]

Figure 3.11 There is a "corner" at $x = 0$.

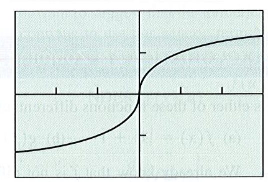


[−3, 3] by [−2, 2]

Figure 3.12 There is a "cusp" at $x = 0$.

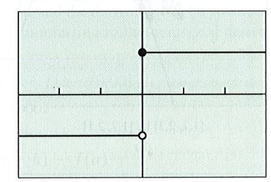
2. a *cusp,* where the slopes of the secant lines approach ∞ from one side and $-\infty$ from the other (an extreme case of a corner); Example: $f(x) = x^{2/3}$

3. a *vertical tangent,* where the slopes of the secant lines approach either ∞ or $-\infty$ from both sides (in this example, ∞); Example: $f(x) = \sqrt[3]{x}$



[−3, 3] by [−2, 2]

Figure 3.13 There is a vertical tangent line at $x = 0$.



[−3, 3] by [−2, 2]

Figure 3.14 There is a discontinuity at $x = 0$.

4. a *discontinuity* (which will cause one or both of the one-sided derivatives to be nonexistent). Example: The *Unit Step Function*

$$U(x) = \begin{cases} -1, & x < 0 \\ 1, & x \ge 0 \end{cases}$$

In this example, the left-hand derivative fails to exist:

$$\lim_{h \to 0^-} \frac{U(0 + h) - U(0)}{h} = \lim_{h \to 0^-} \frac{(-1) - (1)}{h} = \lim_{h \to 0^-} \frac{-2}{h} = \infty.$$